

# GEOMETRIC FUNCTION THEORY FOR CERTAIN SLICE REGULAR FUNCTIONS

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**ABSTRACT.** In this paper, we shall study the geometric function theory for slice regular functions of a quaternionic variable. Specially, we give some coefficient estimates for slice regular functions among which a version of the Bieberbach theorem and the Fekete-Szegő inequality for a subclass of slice regular functions are presented. The growth and distortion theorems for this class are also established. Finally, we give a Bloch theorem for slice regular functions with convex image.

## 1. INTRODUCTION

In one complex variable, the celebrated Bieberbach conjecture was proved by de Branges in 1985 [2], stating that

**Theorem 1.1.** *Let  $f(z) = z + \sum_{n=2}^{+\infty} a_n z^n$  be an injective holomorphic function on the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ , then*

$$|a_n| \leq n, \quad n = 2, 3, \dots$$

*with strict inequality for all  $n$  unless  $f$  is a rotation of the Koebe function.*

Unfortunately, in [3] Cartan pointed out that the above theorem does not hold in several complex variables as shown by a counterexample (cf. [7]). Thus, one needs to restrict the class of mappings to a specific subclass, such as normalized biholomorphic starlike mappings or convex mappings, to obtain a Bieberbach theorem in higher dimensions.

As a generalization of one complex variable, the theory of slice regular functions is initiated by Gentili and Struppa [12, 13]. Many geometric properties of this class of functions have been already studied such as Schwarz-Pick lemma [1], the Bohr theorem [24], the Bloch-Landau theorem [23, 30], Landau-Toeplitz theorem [14] and the Borel-Carathéodory theorem [25]. In this article, we shall continue the study of the geometric property of slice regular functions. Our main results are the quaternionic version of the Bieberbach theorem as well as some well-known coefficient estimates and the growth, distortion and covering theorems for some subclasses of slice regular functions.

Let  $\mathbb{B}$  be the open unit ball in quaternions  $\mathbb{H}$ . Denote by  $\mathbb{S}$  the unit 2-sphere of purely imaginary quaternions. As in the complex holomorphic case, let  $\mathcal{P}$  denote the class of slice regular functions  $p$  on  $\mathbb{B}$  such that  $p(0) = 1$  and  $\operatorname{Re} p(q) > 0$  on  $\mathbb{B}$ . This class is usually called the Carathéodory class. Let us introduce two subclasses of slice regular functions. Let  $\Omega \subset \mathbb{H}$

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2010 *Mathematics Subject Classification.* Primary 30G35; Secondary 30C50.

*Key words and phrases.* Bieberbach conjecture, Fekete-Szegő inequality, growth and distortion theorems, Bloch theorem, quaternions.

This work was supported by the NNSF of China (11071230).

and denote by  $\Omega_I$  the intersection  $\Omega \cap \mathbb{C}_I$ . Denote

$$\mathcal{N}(\mathbb{B}) = \{f : \mathbb{B} \rightarrow \mathbb{H} : f \text{ is slice regular such that } f(\Omega_I) \subseteq \mathbb{C}_I \text{ for all } I \in \mathbb{S}\}.$$

Another subclass of slice regular functions is given by

$$\mathcal{V}(\mathbb{B}) = \{f : \mathbb{B} \rightarrow \mathbb{H} : f \text{ is slice regular such that } f(\Omega_I) \subseteq \mathbb{C}_I \text{ for some } I \in \mathbb{S}\}.$$

According to Theorem 1.1, a quaternionic version of Bieberbach theorem can be established for the normalized and injective function  $f \in \mathcal{V}(\mathbb{B})$ . It is a natural question if  $\mathcal{V}(\mathbb{B})$  is the largest class of injective slice regular functions in which the Bieberbach conjecture holds (see [9]).

In the present paper, we shall answer partially this question and consider a large subclass  $\mathcal{C}$  of slice regular functions. Denote

$$\mathcal{S} = \{f : \mathbb{B} \rightarrow \mathbb{H} : f \text{ is slice regular such that } \mathcal{Z}_f = \{0\} \text{ and } f'(0) = 1\},$$

where  $\mathcal{Z}_f$  denotes the zero set of the function  $f$ .

Now the subclass  $\mathcal{C}$  of slice regular functions is defined by

$$\mathcal{C} = \{f : \mathbb{B} \rightarrow \mathbb{H} : \text{there exists a function } h \in \mathcal{S}^* \text{ such that } \operatorname{Re}(h(q)^{-1}qf'(q)) > 0 \text{ on } \mathbb{B}\},$$

where

$$\mathcal{S}^* = \{f : \mathbb{B} \rightarrow \mathbb{H} : f \in \mathcal{S} \text{ such that } \operatorname{Re}(f(q)^{-1}qf'(q)) > 0 \text{ on } \mathbb{B}\}.$$

Note that the inclusion  $\mathcal{S}^* \subset \mathcal{C}$  holds. Let us say more about the function class  $\mathcal{S}^*$ . To this end, we recall the definition of starlike function.

Let  $\Omega \subset \mathbb{R}^n$ . The set  $\Omega$  is called starlike with respect to the origin 0, if  $t\Omega \subset \Omega$  for all  $t \in [0, 1]$ .  $\Omega$  is called convex if  $(1-t)\Omega + t\Omega \subset \Omega$  for all  $t \in [0, 1]$ . Let  $F$  be holomorphic on  $\mathbb{D}$ . We say that  $F$  is a starlike (resp. convex) function on  $\mathbb{D}$  with respect to 0 if  $F$  is injective on  $\mathbb{D}$  and the image  $F(\mathbb{D})$  is starlike (resp. convex) with respect to 0.

In one complex variable case, a holomorphic function  $f$  in  $\mathcal{S}^*$  is exactly the starlike function (see Theorem 3.2). Hence, the result obtained in this paper is a generalization from starlike functions to the function class  $\mathcal{S}^*$  in the noncommutative algebra  $\mathbb{H}$ .

In fact, the algebraic condition in  $\mathcal{S}^*$  can be described as a geometric restriction that the element in  $\mathcal{S}^*$  is a slice regular function such that its modulus is strictly increasing in the radial. However, we do not know if the image  $f(\mathbb{B})$  is starlike for any  $f \in \mathcal{S}^*$ .

We are now in a position to state our first main result as follows.

**Theorem 1.2.** *Let  $f(q) = q + \sum_{n=2}^{+\infty} q^n a_n \in \mathcal{C}$ . Then*

$$|a_n| \leq n, \quad \text{for all } n = 2, 3, \dots$$

*Equality  $|a_n| = n$  for a given  $n \geq 2$  holds if and only if  $f$  is of the form*

$$f(q) = q(1 - qu)^{-*2}, \quad \forall q \in \mathbb{B}$$

*for some  $u \in \partial\mathbb{B}$ .*

In [6] Fekete and Szegő proved a striking inequality, related to the Bieberbach conjecture,

$$(1.1) \quad |a_3 - \lambda a_2^2| \leq \begin{cases} 3 - 4\lambda, & \text{if } \lambda \in (-\infty, 0), \\ 1 + 2e^{-2\lambda/(1-\lambda)}, & \text{if } \lambda \in [0, 1), \\ 4\lambda - 3, & \text{if } \lambda \in [1, +\infty), \end{cases}$$

for any injective holomorphic function  $f(z) = z + \sum_{n=2}^{+\infty} a_n z^n$  on  $\mathbb{D}$ . Moreover, inequality (1.1) is sharp for each  $\lambda$ . For related complex versions of Theorem 1.3, we refer the reader to the literature [19, 20] and [5]. In this paper, we shall consider Fekete-Szegő type inequality for slice regular functions as follows.

**Theorem 1.3.** *Let  $f(q) = q + \sum_{n=2}^{+\infty} q^n a_n \in \mathcal{S}^*$ . Then, for any  $\lambda \in \mathbb{H}$ ,*

$$|a_3 - \lambda a_2^2| \leq \max\{1, |4\lambda - 3|\}.$$

*This estimate is sharp for each  $\lambda$ . Equality occurs if*

$$f(q) = q(1 - qu)^{-*2}, \quad q \in \mathbb{B},$$

*for any  $u \in \partial\mathbb{B}$ .*

Note that if  $\lambda = 0$  in Theorem 1.3, then this result is already obtained in Theorem 1.2.

In geometric function theory of holomorphic functions of one complex variable, the well-known growth and distortion theorems play an important role in the systematic study of univalent functions saying that

**Theorem 1.4.** *Let  $F$  be an injective holomorphic function on  $\mathbb{D}$  such that  $F(0) = 0$  and  $F'(0) = 1$ . Then for each  $z \in \mathbb{D}$ , the following inequalities hold:*

$$(1.2) \quad \frac{|z|}{(1 + |z|)^2} \leq |F(z)| \leq \frac{|z|}{(1 - |z|)^2};$$

$$(1.3) \quad \frac{1 - |z|}{(1 + |z|)^3} \leq |F'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3};$$

$$(1.4) \quad \frac{1 - |z|}{1 + |z|} \leq \left| \frac{zF'(z)}{F(z)} \right| \leq \frac{1 + |z|}{1 - |z|}.$$

*Moreover, equality holds for one of these six inequalities at some  $z_0 \in \mathbb{D} \setminus \{0\}$  if and only if*

$$F(z) = \frac{z}{(1 - e^{i\theta}z)^2}, \quad \forall z \in \mathbb{D},$$

*for some  $\theta \in \mathbb{R}$ .*

However, the growth and distortion theorems fail generally in  $\mathbb{C}^n$  ( $n \geq 2$ ) and only hold for some subclass of starlike or convex mappings (cf. [11, 8]). As a direct result of Theorem 1.4, the growth, distortion and covering theorems can be obtained for normalized injective slice regular functions in the special class  $\mathcal{N}(\mathbb{B})$  (see [9, Theorem 3.11]). Recently, the authors in [26] have generalized this result to normalized injective (on  $\mathbb{B}_I$ ) slice regular functions in  $\mathcal{V}(\mathbb{B})$  and raised the question if the class  $\mathcal{V}(\mathbb{B})$  is the largest subclass of slice regular functions in which the corresponding growth, distortion and covering theorems hold.

In this paper, we give a positive result on growth, distortion and covering theorems for slice regular functions in the larger subclass  $\mathcal{S}^*$  in Section 5, though we are still unable to solve the general case.

The close analytic connection between convex and starlike function known as Alexander's theorem asserts that  $F$  is convex if and only if  $zF'(z)$  is starlike for the holomorphic function  $F$  on  $\mathbb{D}$  with  $F(0) = 0$  and  $F'(0) = 1$ . In view of this result, we give another coefficient estimate for a subclass of slice regular functions from Theorem 1.2.

**Theorem 1.5.** *Let  $f(q) = q + \sum_{n=2}^{+\infty} q^n a_n$  be a slice regular function on  $\mathbb{B}$  such that  $qf'(q) \in \mathcal{S}^*$ . Then*

$$|a_n| \leq 1, \quad \text{for all } n = 2, 3, \dots$$

*Equality  $|a_n| = 1$  for a given  $n \geq 2$  holds if and only if  $f$  is of the form*

$$f(q) = q(1 - qu)^{-*}, \quad \forall q \in \mathbb{B}$$

*for some  $u \in \partial\mathbb{B}$ .*

Theorem 1.5 directly implies the following growth theorem:

$$|f(q)| \leq \frac{|q|}{1 - |q|}, \quad \forall q \in \mathbb{B},$$

for any slice regular function  $f$  on  $\mathbb{B}$  such that  $f(0) = 0$  and  $qf'(q) \in \mathcal{S}^*$ .

For normalized convex functions  $F$  on  $\mathbb{D}$ , there holds a sharper growth theorem

$$(1.5) \quad \frac{|z|}{1 + |z|} \leq |F(z)| \leq \frac{|z|}{1 - |z|}, \quad \forall z \in \mathbb{D}.$$

which was generalized to  $\mathbb{C}^n$  (see e.g. [29, 21]).

Usually, inequality (1.5) is just a consequence of the following distortion theorem in the complex variable case.

**Theorem 1.6.** *Let  $f$  be a slice regular function on  $\mathbb{B}$  such that  $qf'(q) \in \mathcal{S}^*$ . Then*

$$\frac{1}{(1 + |q|)^2} \leq |f'(q)| \leq \frac{1}{(1 - |q|)^2}, \quad \forall q \in \mathbb{B}.$$

*Estimates are sharp. For each  $q \in \mathbb{B} \setminus \{0\}$ , equality occurs if*

$$f(q) = q(1 - qu)^{-*}, \quad q \in \mathbb{B},$$

*for some  $u \in \partial\mathbb{B}$ .*

Unfortunately, the method adopted in one complex variable can not be used to our setting due to the higher dimension and we do not know if the image of  $qf'(q)$  is convex for any  $f \in \mathcal{S}^*$ . However, we can obtain a Koebe type theorem for convex slice regular functions of a quaternionic variable as follows.

**Theorem 1.7.** *Let  $f$  be a slice regular function on  $\mathbb{B}$  with convex image and  $f'(0) = 1$ . Then  $f(\mathbb{B})$  contains a open ball centred at  $f(0)$  of radius  $1/2$ . The constant  $1/2$  is optimal.*

The remaining part of this paper is organized as follows.

In Section 2, we set up basic notation and give some preliminary results from the theory of slice regular functions over quaternions.

In Section 3, we first offer some examples of  $\mathcal{C}$  and establish some useful lemmas to prove Theorems 1.2 and 1.3 among which coefficient estimates for the Carathéodory class in the quaternionic setting are established. Besides, we give some equivalent descriptions of the class  $\mathcal{S}^*$  (see Lemma 3.7) which allow us to give a new characterization of starlikeness in the one complex variable case (see Corollary 3.8) and see that the so-called slice-starlike in [10, Definition 3.17] and algebraically starlike in [16, Definition 5.20] are equivalent. We also recall a Schwarz lemma from [13] and give a Rogosinski lemma for slice regular functions.

In Section 4, Theorems 1.2 and 1.3 are proved.

In Section 5, we shall establish the growth and distortion theorems for some classes of slice regular functions in the quaternionic setting corresponding to Theorem 1.4. As an application, the Koebe one-quarter theorem for the class  $\mathcal{S}^*$  is obtained. Moreover, we formulate a result of Hayman (see Theorem 5.5) which is a sharper version of growth theorem. Finally, we prove Theorem 1.7 which allows to establish the Bloch theorem for convex slice regular functions.

It is worth mentioning that the regularity does not keep under usual product and composition of two slice regular functions due to the non-commutativity of quaternions. Hence, many methods in complex variable are may not valid for the quaternionic setting. For example, the subordination argument in [17] is not suitable to Theorem 5.2. In addition, the non-commutativity of quaternions leads to a complicated computation in theorems and forces us to formulate some new auxiliary functions in the proof of Theorem 3.13 and to explore other methods such as in Theorems 5.1 and 1.7.

## 2. PRELIMINARIES

In this section we recall some necessary definitions and preliminary results on slice regular functions. To have a more complete insight on the theory, we refer the reader to the monograph [15, 4].

Let  $\mathbb{H}$  denote the non-commutative, associative, real algebra of quaternions with standard basis  $\{1, i, j, k\}$ , subject to the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

Every element  $q = x_0 + x_1i + x_2j + x_3k$  in  $\mathbb{H}$  is composed by the *real* part  $\operatorname{Re}(q) = x_0$  and the *imaginary* part  $\operatorname{Im}(q) = x_1i + x_2j + x_3k$ . The *conjugate* of  $q \in \mathbb{H}$  is then  $\bar{q} = \operatorname{Re}(q) - \operatorname{Im}(q)$  and its *modulus* is defined by  $|q|^2 = q\bar{q} = |\operatorname{Re}(q)|^2 + |\operatorname{Im}(q)|^2$ . We can therefore calculate the multiplicative inverse of each  $q \neq 0$  as  $q^{-1} = |q|^{-2}\bar{q}$ . Every  $q \in \mathbb{H}$  can be expressed as  $q = x + yI$ , where  $x, y \in \mathbb{R}$  and

$$I = \frac{\operatorname{Im}(q)}{|\operatorname{Im}(q)|}$$

if  $\operatorname{Im} q \neq 0$ , otherwise we take  $I$  arbitrarily such that  $I^2 = -1$ . Then  $I$  is an element of the unit 2-sphere of purely imaginary quaternions

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}.$$

For every  $I \in \mathbb{S}$  we will denote by  $\mathbb{C}_I$  the plane  $\mathbb{R} \oplus I\mathbb{R}$ , isomorphic to  $\mathbb{C}$ , and, if  $\Omega \subset \mathbb{H}$ , by  $\Omega_I$  the intersection  $\Omega \cap \mathbb{C}_I$ .

We can now recall the definition of slice regularity.

**Definition 2.1.** Let  $\Omega$  be a domain in  $\mathbb{H}$ . A function  $f : \Omega \rightarrow \mathbb{H}$  is called (left) *slice regular* if, for all  $I \in \mathbb{S}$ , its restriction  $f_I$  to  $\Omega_I$  is *holomorphic*, i.e., it has continuous partial derivatives and satisfies

$$\bar{\partial}_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0$$

for all  $x + yI \in \Omega_I$ .

**Definition 2.2.** Let  $f : \mathbb{B} \rightarrow \mathbb{H}$  be a slice regular function. For each  $I \in \mathbb{S}$ , the *I-derivative* of  $f$  at  $q = x + yI$  is defined by

$$\partial_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI)$$

on  $\mathbb{B}_I$ . The *slice derivative* of  $f$  is the function  $f'$  defined by  $\partial_I f$  on  $\mathbb{B}_I$  for all  $I \in \mathbb{S}$ .

All slice regular functions on  $\mathbb{B}$  can be expressed as power series.

**Theorem 2.3.** *Let  $f : \mathbb{B} \rightarrow \mathbb{H}$  be a slice regular function. Then*

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \quad \text{with } a_n = \frac{f^{(n)}(0)}{n!}$$

for all  $q \in \mathbb{B}$ .

**Definition 2.4.** Let  $f, g : \mathbb{B} \rightarrow \mathbb{H}$  be two slice regular functions of the form

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \quad g(q) = \sum_{n=0}^{\infty} q^n b_n.$$

The regular product (or  $*$ -product) of  $f$  and  $g$  is the slice regular function defined by

$$f * g(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^n a_k b_{n-k} \right).$$

**Definition 2.5.** Let  $f(q) = \sum_{n=0}^{\infty} q^n a_n$  be a slice regular function on  $\mathbb{B}$ . We define the *regular conjugate* of  $f$  as

$$f^c(q) = \sum_{n=0}^{\infty} q^n \bar{a}_n,$$

and the *symmetrization* of  $f$  as

$$f^s(q) = f * f^c(q) = f^c * f(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^n a_k \bar{a}_{n-k} \right).$$

Both  $f^c$  and  $f^s$  are slice regular functions on  $\mathbb{B}$ .

Let  $\mathcal{Z}_{f^s}$  denote the zero set of the symmetrization  $f^s$  of  $f$ .

**Definition 2.6.** Let  $f$  be a slice regular function on  $\mathbb{B}$ . If  $f$  does not vanish identically, its *regular reciprocal* is the function defined by

$$f^{-*}(q) := f^s(q)^{-1} f^c(q)$$

and it is slice regular on  $\mathbb{B} \setminus \mathcal{Z}_{f^s}$ .

**Proposition 2.7.** *Let  $f$  and  $g$  be slice regular functions on  $\mathbb{B}$ . Then for all  $q \in \mathbb{B} \setminus \mathcal{Z}_{f^s}$ ,*

$$f^{-*} * g(q) = f(T_f(q))^{-1} g(T_f(q)),$$

where  $T_f : \mathbb{B} \setminus \mathcal{Z}_{f^s} \rightarrow \mathbb{B} \setminus \mathcal{Z}_{f^s}$  is defined by  $T_f(q) = f^c(q)^{-1} q f^c(q)$ . Furthermore,  $T_f$  and  $T_{f^c}$  are mutual inverses so that  $T_f$  is a diffeomorphism.

Note that we have omitted the word “left” in the article for simplicity. Certainly, the right slice regular functions have completely analogous theories. This point shall be used in the proof of Theorem 3.13.

## 3. EXAMPLES AND LEMMAS

In this section, we shall offer some examples in the function class  $\mathcal{C}$  and give some useful lemmas to prove Theorems 1.2 and 1.3. In addition, we establish a Rogosinski lemma for slice regular functions which is a sharpened form of the Schwarz lemma.

**Example 3.1.** Let  $f(q) = q + \sum_{n=2}^{+\infty} q^n a_n$  be a slice regular function on  $\mathbb{B}$  such that  $\sum_{n=2}^{+\infty} n|a_n| \leq 1$ . Then  $f \in \mathcal{S}^*$ .

*Proof.* By the assumption of  $f$ , we have

$$|qf'(q) - f(q)| \leq \sum_{n=2}^{+\infty} |q|^n (n-1) |a_n| < |q| - \sum_{n=2}^{+\infty} |q|^n |a_n| \leq |f(q)|, \quad \forall q \in \mathbb{B} \setminus \{0\}.$$

Hence,  $\mathcal{Z}_f = \{0\}$  and

$$|f(q)^{-1}qf'(q) - 1| = |f(q)^{-1}| |qf'(q) - f(q)| < 1, \quad \forall q \in \mathbb{B} \setminus \{0\},$$

which also implies that  $\operatorname{Re}(f(q)^{-1}qf'(q)) > 0$  on  $\mathbb{B}$ , as desired.  $\square$

In order to introduce another important class of examples, we need recall a well-known analytical characterization of starlikeness (see e.g. [11, p. 36]).

**Theorem 3.2.** Let  $F : \mathbb{D} \rightarrow \mathbb{C}$  be a holomorphic function with  $F(0) = 0$  and  $F'(0) = 1$ . Then  $F$  is starlike with respect to 0 if and only if  $\operatorname{Re} \frac{zF'(z)}{F(z)} > 0$  on  $\mathbb{D}$ .

We recall also a so-called convex combination identity for slice regular functions which was used to establish sharp growth and distortion theorems for a subclass of slice regular functions (see [26] or [27]).

**Lemma 3.3.** Let  $f$  be a slice regular function on  $\mathbb{B}$  such that  $f(\mathbb{B}_I) \subseteq \mathbb{C}_I$  for some  $I \in \mathbb{S}$ . Then for every  $re^{J\theta} \in \mathbb{B}$  with  $J \in \mathbb{S}$ ,

$$(3.1) \quad |f(re^{J\theta})|^2 = \frac{1 + \langle I, J \rangle}{2} |f(re^{I\theta})|^2 + \frac{1 - \langle I, J \rangle}{2} |f(re^{-I\theta})|^2$$

where  $\langle I, J \rangle = -\operatorname{Re}(IJ) \in (-1, 1)$ .

From Theorem 3.2, Lemmas 3.3 and 3.7 below, one can show that

**Example 3.4.** Fix  $I \in \mathbb{S}$ . Let  $F(z) = z + \sum_{n=2}^{+\infty} z^n a_n$ ,  $a_n \in \mathbb{C}_I$  for  $n = 2, 3, 4, \dots$  be a starlike function on  $\mathbb{B}_I \simeq \mathbb{D}$ . Then  $f(q) = q + \sum_{n=2}^{+\infty} q^n a_n \in \mathcal{S}^*$ .

**Example 3.5.** Let  $f \in \mathcal{S}$  such that

$$(3.2) \quad \operatorname{Re}((f(q) - f(-q))^{-1}qf'(q)) > 0, \quad \forall q \in \mathbb{B}.$$

Then  $f \in \mathcal{C}$ .

*Proof.* From (3.6) below, it follow that

$$r \frac{\partial}{\partial r} \log |f(q) - f(-q)| = \operatorname{Re} \left( (f(q) - f(-q))^{-1} q(f'(q) + f'(-q)) \right),$$

for  $q = ru \in \mathbb{B} \setminus \{0\}$  with  $r = |q|$ .

Then condition (3.2) and Lemma 3.7 imply that  $(f(q) - f(-q))/2 \in \mathcal{S}^*$ , thus  $f \in \mathcal{C}$  by definition.  $\square$

To show Lemma 3.7, we prove

**Proposition 3.6.** *Let  $f, g$  be two slice regular functions on  $\mathbb{B}$  such that  $\mathcal{Z}_f = \emptyset$ . Then we have*

$$(3.3) \quad \operatorname{Re}(f(q)^{-1}g(q)) > 0 \text{ on } \mathbb{B} \Leftrightarrow \operatorname{Re}(f(q)^{-*} * g(q)) > 0 \text{ on } \mathbb{B},$$

and

$$(3.4) \quad |g(q)| < |f(q)| \text{ on } \mathbb{B} \Leftrightarrow |f(q)^{-*} * g(q)| < 1 \text{ on } \mathbb{B}.$$

*Proof.* We only prove the first claim (3.3). The second can be obtained by the same strategy. Note that  $\mathcal{Z}_f = \emptyset$  if and only if  $\mathcal{Z}_{f^c} = \emptyset$  for any slice regular function on  $\mathbb{B}$ . Assume that  $\operatorname{Re}(f(q)^{-1}g(q)) > 0$  on  $\mathbb{B}$ . From the assumption on the functions  $f$  and  $g$ , we have that, by Proposition 2.7,

$$(3.5) \quad f^{-*} * g(q) = f(T_f(q))^{-1}g(T_f(q)), \quad \forall q \in \mathbb{B},$$

where  $T_f(q) = f^c(q)^{-1}qf^c(q) \in \mathbb{B}$ . Hence, it holds that  $\operatorname{Re}(f(q)^{-*} * g(q)) > 0$  on  $\mathbb{B}$ .

Conversely, the condition  $\operatorname{Re}(f(q)^{-*} * g(q)) > 0$  on  $\mathbb{B}$  implies that  $\operatorname{Re}(f(T_f(q))^{-1}g(T_f(q))) > 0$  on  $\mathbb{B}$  by (3.5). Then the desired result follows from the fact that  $T_f \circ T_{f^c}(q) = q$  for all  $q \in \mathbb{B}$ .  $\square$

In one complex variable, the starlike and convex functions of order  $\alpha$  were first introduced by Robertson [22]. Now we introduce its corresponding version for slice regular functions. Let  $\alpha < 1$ . Denote

$$\mathcal{S}^*(\alpha) = \{f : \mathbb{B} \rightarrow \mathbb{H} : f \in \mathcal{S} \text{ such that } \operatorname{Re}(f(q)^{-1}qf'(q)) > \alpha \text{ on } \mathbb{B}\}.$$

**Lemma 3.7.** *Let  $f$  be a slice regular function on  $\mathbb{B}$  such that  $\mathcal{Z}_f = \{0\}$  and  $f'(0) = 1$ . Then, for  $\alpha < 1$ , the following statements are equivalent:*

- (1)  $\operatorname{Re}(f(q)^{-1}qf'(q)) > \alpha$  on  $\mathbb{B} \setminus \{0\}$ ;
- (2)  $\operatorname{Re}(f(q)^{-*} * (qf'(q))) > \alpha$  on  $\mathbb{B} \setminus \{0\}$ ;
- (3) For any  $u \in \partial\mathbb{B}$ , the function  $M(r) = |f(ru)|/r^\alpha$  is strictly increasing on  $(0, 1)$ .

*Proof.* (1)  $\Leftrightarrow$  (2): It is trivial by Proposition 3.6.

(1)  $\Rightarrow$  (3): Notice that, for any slice regular function  $f(q)$  on  $\mathbb{B}$  with  $q = ru = |q|u \in \mathbb{B}$ , there holds that

$$r \frac{\partial}{\partial r} f(q) = qf'(q).$$

Then, for  $q \in \mathbb{B} \setminus \{0\}$ ,

$$(3.6) \quad r \frac{\partial}{\partial r} \log |f(q)| = \frac{r}{|f(q)|^2} \operatorname{Re} \left( \frac{\partial}{\partial r} f(q) \overline{f(q)} \right) = \operatorname{Re}(f(q)^{-1}qf'(q)) > \alpha.$$

Integrating the above inequality from  $r_1$  to  $r_2$  with  $0 < r_1 < r_2 < 1$  gives that

$$\frac{|f(r_2u)|}{|f(r_1u)|} > \frac{r_2^\alpha}{r_1^\alpha},$$

which implies statement (3).

(3)  $\Rightarrow$  (1): Fixe  $u \in \partial\mathbb{B}$ . Under the condition of (3), we have, for  $q = ru, r \in (0, 1)$ ,

$$\frac{\partial}{\partial r} \frac{|f(ru)|}{r^\alpha} \geq 0,$$

which implies

$$r \frac{\partial}{\partial r} \log |f(ru)| \geq \alpha.$$



This together with (3.6) shows that  $\operatorname{Re}(f(q)^{-1}qf'(q)) \geq \alpha$  for all  $r \in (0, 1)$ . Since  $u \in \partial\mathbb{B}$  is arbitrary, it follows that

$$(3.7) \quad \operatorname{Re}(f(q)^{-1}qf'(q)) \geq \alpha, \quad \forall q \in \mathbb{B} \setminus \{0\}.$$

Let us show inequality (3.7) is strict. Consider the slice regular function

$$h(q) = \begin{cases} f(q)^{-*} * (qf'(q)) & \text{if } q \in \mathbb{B} \setminus \{0\}, \\ 1, & \text{if } q = 0. \end{cases}$$

By Proposition 3.6, (3.7) implies that

$$\operatorname{Re} h(q) \geq \alpha, \quad \forall q \in \mathbb{B}.$$

If equality occurs in (3.7) for some  $q_0 \in \mathbb{B}$ , then  $\operatorname{Re} h(\tilde{q}_0) = \alpha$  for  $\tilde{q}_0 = f(q_0)^{-1}q_0f(q_0) \in \mathbb{B}$ . By applying the maximum principle (see [15, Theorem 7.13]) for the real part of a slice regular function to  $-h$ , we see that  $h$  is constant on  $\mathbb{B}$ . Whence  $f(q) = qf'(q)$  on  $\mathbb{B}$ , which forces that  $f$  is an identity on  $\mathbb{B}$ . Thus, for  $\alpha < 1$ , (3.7) is strict.

Hence statements (1) and (3) are equivalent.  $\square$

Note that  $\mathcal{S}^*(\alpha) = \emptyset$  if  $\alpha > 1$  from Lemma 3.7.

As a byproduct, it follows, by Theorem 3.2 and Lemma 3.7, that

**Corollary 3.8.** *Let  $F$  be holomorphic on  $\mathbb{D}$  such that  $F(0) = 0$  and  $F'(0) = 1$ . Then the following statements are equivalent.*

- (1)  $F$  is starlike with respect to 0;
- (2)  $\operatorname{Re}(F(z)^{-1}zF'(z)) > 0$  on  $\mathbb{D}$ ;
- (3) For any  $\theta \in [0, 2\pi)$ , the function  $M(r) = |F(re^{i\theta})|$  is strictly increasing on  $[0, 1)$ .

**Remark 3.9.** Note that condition (2) in Corollary 3.8 would imply that  $F$  is injective on  $\mathbb{D}$ . From the formula

$$\operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) = \frac{\partial}{\partial\theta} \arg F(re^{i\theta}), \quad z = re^{i\theta},$$

it follows that the argument of  $F(re^{i\theta})$  increases monotonically for  $\theta$ . Thus  $F$  is injective on  $|z| = r < 1$ . By the principle of univalence on the boundary, we can conclude that  $F$  is injective on  $|z| \leq r$ . Since  $r < 1$  is arbitrary,  $F$  is injective on  $\mathbb{D}$  (see e.g. [11, p. 38]).

However, the above approach is not suitable for the quaternionic setting and we pose a natural unsolved question:

*Is any slice regular function  $f$  such that  $f(0) = 0$ ,  $f'(0) = 1$  and  $\operatorname{Re}(f(q)^{-1}qf'(q)) > 0$  on  $\mathbb{B}$  injective?*

Now we state without proof a variant of the Schwarz lemma for slice regular functions in [13, Theorem 4.1] which is useful in the sequel.

**Theorem 3.10.** *If  $f : \mathbb{B} \rightarrow \mathbb{B}$  be slice regular such that  $f(0) = f'(0) = \dots = f^{(m-1)}(0) = 0$  ( $m \geq 1$ ). Then*

$$|f(q)| \leq |q|^m, \quad \forall q \in \mathbb{B},$$

and

$$|f^{(m)}(0)| \leq m!.$$

Both inequalities are strict (except  $q = 0$ ) unless  $f(q) = q^m u$  for some  $u \in \partial\mathbb{B}$ .

The classical Schwarz-Pick lemma states that if  $F$  is a holomorphic self-mapping on  $\mathbb{D}$ , then

$$(3.8) \quad |F'(z)| \leq \frac{1 - |F(z)|^2}{1 - |z|^2}, \quad \forall z \in \mathbb{D}.$$

However, this classical version fails even for the slice regular automorphisms. Indeed, we take the slice regular Möbius transformation (see [28, Theorem 7.6])

$$\varphi_a(q) = (1 - q\bar{a})^{-*} * (a - q) = a - (1 - |a|^2) \sum_{n=1}^{+\infty} q^n \bar{a}^{n-1}.$$

Pick  $a = \frac{i}{2}$  and  $q_0 = \frac{j}{2}$ . By direct computation, we have

$$\varphi_a(q_0) = \frac{2}{5}(i - j), \quad \varphi'_a(q_0) = -\frac{204}{225} - \frac{96}{225}k,$$

so that

$$|\varphi'_a(q_0)| = \sqrt{\frac{50832}{50625}} > \frac{1 - |\varphi_a(q_0)|^2}{1 - |q_0|^2} = \frac{68}{75}.$$

Fortunately, a version of the Schwarz-Pick lemma for self-mapping of  $\mathbb{B}$  was established in [1].

As a special case, one can obtain a coefficient estimate.

**Lemma 3.11.** *Let  $f$  be a slice regular function on  $\mathbb{B}$  such that  $|f(q)| \leq 1$  for all  $q \in \mathbb{B}$ . Then*

$$(3.9) \quad |f'(0)| \leq 1 - |f(0)|^2.$$

*Equality holds in (3.9) if and only if  $f$  is of the form*

$$f(q) = (1 - q\bar{a})^{-*} * (a - q)u, \quad \forall q \in \mathbb{B},$$

*for some  $a \in \overline{\mathbb{B}}, u \in \partial\mathbb{B}$ .*

**Remark 3.12.** Based on Lemma 3.11, the authors in [24] established the quaternionic analog of the sharp version of the Bohr Theorem saying that

Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  be a slice regular function on  $\mathbb{B}$ , continuous on the closure  $\overline{\mathbb{B}}$ , such that  $|f(q)| < 1$  for all  $|q| \leq 1$ . Then

$$\sum_{n=0}^{+\infty} |q^n a_n| < 1, \quad |q| \leq \frac{1}{3}.$$

Moreover  $1/3$  is the largest radius for which the statement is true.

By applying coefficient estimates (7) in [25, Theorem 4], we can give the Bohr Theorem in a more general setting as follows.

Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  be a slice regular function on  $\mathbb{B}$  such that  $\operatorname{Re} f(q) \leq 1$  for all  $|q| < 1$ . Then

$$\sum_{n=0}^{+\infty} |q^n a_n| < 1, \quad |q| < \frac{1}{3}.$$

Now we establish a sharpened form of the Schwarz lemma for slice regular functions. Its complex version is due to Rogosinski (see e.g. [5, p. 200]).

**Theorem 3.13.** *Let  $q_0, b \in \mathbb{B}$ . For the set of all slice regular functions  $f : \mathbb{B} \rightarrow \mathbb{B}$  with  $f(0) = 0$  and  $f'(0) = b$ , the range of values of  $f(q_0)$  is the closed ball  $\overline{B(c, r)}$ , where*

$$c = \frac{q_0 b(1 - |q_0|^2)}{1 - |q_0 b|^2}, \quad r = \frac{|q_0|^2(1 - |b|^2)}{1 - |q_0 b|^2}.$$

*Proof.* For  $b = 0$ , the assertion is trivial. Now we consider the case for  $b \neq 0$ . Denote the slice regular function  $g(q) = q^{-1}f(q)$  on  $\mathbb{B}$  with  $g(0) = b$  and  $|g(q)| \leq 1$  for all  $q \in \mathbb{B}$  by the maximum modulus principle for slice regular functions (cf. [15, Theorem 7.1]). Denote the slice regular function

$$h(q) = (1 - g(q)\bar{b})^{-*} * (b - g(q)),$$

with  $h(0) = 0$  and  $|h(q)| < 1$  for all  $q \in \mathbb{B}$  by Proposition 2.7. From Theorem 3.10, it follows that

$$|h(q)| \leq |q|, \quad \forall q \in \mathbb{B}.$$

That is, by Proposition 2.7,

$$|(1 - g \circ T_{(1-g\bar{b})^c}(q)\bar{b})^{-1}(b - g \circ T_{(1-g\bar{b})^c}(q))| \leq |q|, \quad \forall q \in \mathbb{B}.$$

Since  $T_{(1-g\bar{b})^c} \circ T_{1-g\bar{b}}(q) = q$  and  $|T_{1-g\bar{b}}(q)| = |q|$ , we obtain

$$|(1 - g(q)\bar{b})^{-1}(b - g(q))| \leq |q|, \quad \forall q \in \mathbb{B}.$$

Equivalently,

$$\frac{|f(q) - qb|}{|f(q) - q\bar{b}^{-1}|} \leq |bq|, \quad \forall q \in \mathbb{B}.$$

A direct computation shows that

$$\left| f(q) - \frac{qb(1 - |q|^2)}{1 - |qb|^2} \right| \leq \frac{|q|^2(1 - |b|^2)}{1 - |qb|^2}, \quad \forall q \in \mathbb{B}.$$

Hence  $f(q_0) \in \overline{B(c, r)}$  with  $c$  and  $r$  as desired.

To see that  $\overline{B(c, r)}$  is covered, let  $p \in \overline{\mathbb{B}}$  and set the slice regular function

$$f(q) = q(1 - q|b|p)^{-*} * (|b| - qp) \frac{b}{|b|}, \quad \forall q \in \mathbb{B}.$$

It is easy to see that  $f(0) = 0$  and  $f'(0) = b$ . Now we show that  $|f(q)| < 1$  for all  $q \in \mathbb{B}$ . To this end, we only prove that

$$(3.10) \quad |(1 - q|b|p)^{-*} * (|b| - qp)| < 1, \quad \forall q \in \mathbb{B}.$$

Note that

$$(1 - |b|^2)(1 - |qp|^2) > 0, \quad \forall q \in \mathbb{B},$$

that is to say

$$||b| - qp| < |1 - q|b|p|, \quad \forall q \in \mathbb{B}.$$

Therefore, (3.10) holds by Proposition 3.6. After a simple calculation, there holds that

$$f(q) = \frac{qb(1 - |q|^2)}{1 - |qb|^2} + \frac{q^2(1 - |b|^2)}{1 - |qb|^2} \phi(p) \frac{b}{|b|},$$

where

$$\phi(p) = \bar{q}|b| - (1 - |qb|^2) \sum_{n=1}^{+\infty} (|b|q)^{n-1} p^n = (|b|\bar{q} - p) *_r (1 - q|b|p)^{-*r}$$

is a right slice regular Möbius transform of  $\mathbb{B}$ .

Since  $p \in \overline{\mathbb{B}}$  is arbitrary, it follows that  $\phi(\overline{\mathbb{B}}) = \overline{\mathbb{B}}$ .

Note that, for any  $u \in \partial\mathbb{B}$ ,

$$\{qu : q \in \overline{\mathbb{B}}\} = \overline{\mathbb{B}}.$$

Hence the closed ball  $\overline{B(c, r)}$  is covered. The proof is complete.  $\square$

## 4. PROOF OF THEOREMS 1.2 AND 1.3

To prove Theorems 1.2 and 1.3, we shall present a coefficient estimate for the function class  $\mathcal{P}$  based on Lemma 3.11.

**Theorem 4.1.** *Let  $f(q) = 1 + \sum_{n=1}^{\infty} q^n a_n$  be a function in  $\mathcal{P}$ . Then*

$$(4.1) \quad \left| a_2 - \frac{a_1^2}{2} \right| \leq 2 - \frac{|a_1|^2}{2}.$$

*Equality holds in (4.1) if and only if*

$$f(q) = (q\varphi_a(q)u + 1) * (1 - q\varphi_a(q)u)^{-*},$$

*for some  $a \in \overline{\mathbb{B}}$  and  $u \in \partial\mathbb{B}$ .*

*Proof.* Define

$$g(q) = \begin{cases} q^{-1}(f(q) + 1)^{-*} * (f(q) - 1), & \text{if } q \in \mathbb{B} \setminus \{0\}, \\ a_1/2, & \text{if } q = 0. \end{cases}$$

From the assumption of  $f$ , we have

$$|(f(q) + 1)^{-1}(f(q) - 1)| < 1, \quad \forall q \in \mathbb{B},$$

then, by Proposition 3.6, the function  $h(q) = (f(q) + 1)^{-*} * (f(q) - 1)$  is slice regular on  $\mathbb{B}$  with  $h(0) = 0$ ,  $h'(0) = a_1/2$  and  $|h(q)| < 1$  for all  $q \in \mathbb{B}$ . By the maximum modulus principle for slice regular functions (cf. [15, Theorem 7.1]), we obtain that the function  $g$  is a slice regular function on  $\mathbb{B}$  with  $g'(0) = a_2/2 - a_1^2/4$  and  $|g(q)| \leq 1$  for all  $q \in \mathbb{B}$ . From Lemma 3.11, (4.1) follows and equality holds in (4.1) if and only if  $g(q) = \varphi_a(q)u$  for some  $a \in \overline{\mathbb{B}}$  and  $u \in \partial\mathbb{B}$ . By simple calculations, we deduce that

$$f(q) = (q\varphi_a(q)u + 1) * (1 - q\varphi_a(q)u)^{-*}.$$

Now the proof is complete.  $\square$

As a direct result, we obtain all coefficient estimates for the Carathéodory class  $\mathcal{P}$  applying the approach of finite average as in the case of one complex variable. Now let us recall this result in [25].

**Theorem 4.2.** *Let  $f = 1 + \sum_{n=1}^{\infty} q^n a_n$  be a slice regular function in  $\mathcal{P}$ . Then*

$$(4.2) \quad \frac{1 - |q|}{1 + |q|} \leq \operatorname{Re} f(q) \leq |f(q)| \leq \frac{1 + |q|}{1 - |q|}, \quad \forall q \in \mathbb{B},$$

*and*

$$(4.3) \quad |a_n| \leq 2, \quad n = 1, 2, \dots$$

*Moreover,  $|a_1| = 2$  or equality holds for the first or third inequality in (4.2) at some  $q_0 \neq 0$  if and only if*

$$f(q) = (1 - qu)^{-*} * (1 + qu), \quad \forall q \in \mathbb{B},$$

*for some  $u \in \partial\mathbb{B}$ .*

To prove Theorem 1.2, we shall use the following theorem, which generalizes Theorem 2.2.16 in [11] to the noncommutative algebra. Note that Nevanlinna gave Theorem 4.3 for the normalized starlike functions in 1920 (see e.g. [11]).

**Theorem 4.3.** *Let  $f(q) = q + \sum_{n=2}^{+\infty} q^n a_n \in \mathcal{S}^*$ . Then*

$$|a_n| \leq n, \quad \text{for all } n = 2, 3, \dots$$

*Equality  $|a_n| = n$  for a given  $n \geq 2$  holds if and only if  $f$  is of the form*

$$f(q) = q(1 - qu)^{-*2}, \quad \forall q \in \mathbb{B}$$

*for some  $u \in \partial\mathbb{B}$ .*

*Proof.* Define

$$p(q) = \begin{cases} (q^{-1}f(q))^{-*} * f'(q) & \text{if } q \in \mathbb{B} \setminus \{0\}, \\ 1, & \text{if } q = 0, \end{cases}$$

which belongs to  $\mathcal{P}$  by Lemma 3.7. Set  $p(z) = 1 + qp_1 + \dots$ . Applying Theorem 4.2 to function  $p$ , we have  $|p_n| \leq 2$  for all  $n \geq 1$ . Comparing the coefficients in the power series of  $f'(q)$  and  $q^{-1}f(q) * p(q)$ , we have

$$(4.4) \quad na_n = p_{n-1} + a_2 p_{n-2} + \dots + a_{n-1} p_1 + a_n, \quad n = 2, 3, \dots$$

By induction, we can obtain  $|a_n| \leq n$  for all  $n = 2, 3, \dots$

It is easy to see that if  $|a_n| = n$  for a given  $n$ , then the above arguments imply that  $|p_1| = |a_2| = 2$ , and thus, by Theorem 4.2,  $p$  is of the form

$$p(q) = (1 - qu)^{-*} * (1 + qu) = 1 + \sum_{n=1}^{+\infty} q^n 2u^n, \quad \forall q \in \mathbb{B}$$

for some  $u \in \partial\mathbb{B}$ . Hence, by (4.4), we have

$$(n-1)a_n = 2(u^{n-1} + a_2 u^{n-2} + \dots + a_{n-1} u), \quad n = 2, 3, \dots,$$

which leads to that  $a_n = nu^{n-1}$  for all  $n = 2, 3, \dots$ , as desired.  $\square$

Let us now prove our main coefficient estimates.

*Proof of Theorem 1.2.* By definition, there exists a function  $h = q + q^2 h_2 + \dots \in \mathcal{S}^*$  such that

$$(4.5) \quad p(q) = (q^{-1}h(q))^{-*} * f'(q) = 1 + qp_1 + q^2 p_2 + \dots, \quad q \in \mathbb{B},$$

which belongs to  $\mathcal{P}$  by Lemma 3.7.

From (4.5), we have

$$f'(q) = q^{-1}h(q) * p(q), \quad q \in \mathbb{B}.$$

Comparing the coefficients in the power series above, we have

$$(4.6) \quad na_n = p_{n-1} + h_2 p_{n-2} + \dots + h_{n-1} p_1 + h_n, \quad n = 2, 3, \dots$$

Applying Theorem 4.2 to function  $p$ , we obtain that  $|p_n| \leq 2$  for all  $n \geq 1$ . From the assumption of  $h$ , by Theorem 4.3, we have  $|h_n| \leq n$  for all  $n = 2, 3, \dots$ . Combining these two estimates with (4.6), it follows that  $|a_n| \leq n$  for all  $n = 2, 3, \dots$

Following the proof in Theorem 4.3, the condition of equality can be obtained. This completes the proof.  $\square$

*Proof of Theorem 1.3.* Let us consider the function

$$g(q) = \begin{cases} (qf'(q) + f(q))^{-*} * (qf'(q) - f(q)) & \text{if } q \in \mathbb{B} \setminus \{0\}, \\ 0, & \text{if } q = 0. \end{cases}$$

Then it easy to see that  $g$  is slice regular on  $\mathbb{B}$  with

$$g'(0) = \frac{a_2}{2}, \quad \frac{g''(0)}{2} = a_3 - \frac{3}{4}a_2^2.$$

By Lemma 3.7,  $|g(q)| < 1$  for all  $q \in \mathbb{B}$ , which gives that, by (3.9),

$$\left| a_3 - \frac{3}{4}a_2^2 - \lambda \frac{1}{4}a_2^2 \right| \leq \max\{1, |\lambda|\},$$

that is

$$|a_3 - \lambda a_2^2| \leq \max\{1, |4\lambda - 3|\}.$$

The sharpness of the estimate can be easily checked from the function given in the theorem. Now the proof is complete.  $\square$

## 5. GROWTH AND COVERING THEOREMS

In this section, we shall study the growth, distortion and covering theorems for slice regular functions.

**Theorem 5.1.** *Let  $f \in \mathcal{S}^*$ . Then the following inequalities hold for all  $q \in \mathbb{B}$*

$$(5.1) \quad \frac{|q|}{(1+|q|)^2} \leq |f(q)| \leq \frac{|q|}{(1-|q|)^2};$$

$$(5.2) \quad \frac{1-|q|}{(1+|q|)^3} \leq |f'(q)| \leq \frac{1+|q|}{(1-|q|)^3};$$

$$(5.3) \quad \frac{1-|q|}{1+|q|} \leq \frac{|qf'(q)|}{|f(q)|} \leq \frac{1+|q|}{1-|q|}.$$

*All of these estimates are sharp. For each  $q \in \mathbb{B} \setminus \{0\}$ , equality occurs if*

$$f(q) = q(1-qu)^{-*2}, \quad q \in \mathbb{B},$$

*for some  $u \in \partial\mathbb{B}$ .*

*Proof.* From (3.3) in Proposition 3.6 and Theorem 4.2, we can obtain a combined growth and distortion theorem for the function  $f \in \mathcal{S}^*$ :

$$\frac{1-|q|}{1+|q|} \leq \operatorname{Re}\left(f(q)^{-*} * (qf'(q))\right) \leq \left|f(q)^{-*} * (qf'(q))\right| \leq \frac{1+|q|}{1-|q|}, \quad \forall q \in \mathbb{B}.$$

Thus, by Proposition 2.7,

$$(5.4) \quad \frac{1-|q|}{1+|q|} \leq \operatorname{Re}\left(f(q)^{-1} qf'(q)\right) \leq \frac{|qf'(q)|}{|f(q)|} \leq \frac{1+|q|}{1-|q|}, \quad \forall q \in \mathbb{B}$$

which implies two inequalities in (5.3).

From (3.6) and (5.4), we have

$$\frac{1-r}{r(1+r)} \leq \frac{\partial}{\partial r} \log |f(ru)| = \frac{1}{r} \operatorname{Re}\left(f(q)^{-1} qf'(q)\right) \leq \frac{1+r}{r(1-r)},$$

where  $q = ru$  with  $r = |q|$ .

Integrating the above inequality along a radius from  $r_1$  to  $r_2$  with  $0 < r_1 < r_2 < 1$  gives that

$$(5.5) \quad \frac{r_2(1+r_1)^2}{r_1(1+r_2)^2} \leq \frac{|f(r_2u)|}{|f(r_1u)|} \leq \frac{r_2(1-r_1)^2}{r_1(1-r_2)^2},$$

equivalently,

$$\frac{r_2(1+r_1)^2}{r_1(1+r_2)^2} |f(r_1u)| \leq |f(r_2u)| \leq \frac{r_2(1-r_1)^2}{r_1(1-r_2)^2} |f(r_1u)|.$$

Letting  $r_1 \rightarrow 0^+$ , (5.1) follows from

$$\frac{r_2}{(1+r_2)^2} \leq |f(r_2u)| \leq \frac{r_2}{(1-r_2)^2}.$$

Now (5.2) is a consequence of (5.1) and (5.3). The sharpness of all of these estimates can be checked easily from the function given in the theorem. The proof is complete.  $\square$

Theorem 1.6 is just a consequence of (5.1). Now we establish a general version as follows.

**Theorem 5.2.** *Let  $m$  be a positive integer and  $f(q) = q + \sum_{n=m+1}^{+\infty} q^n a_n$  be a slice regular function on  $\mathbb{B}$  such that  $qf'(q) \in \mathcal{S}^*$ . Then*

$$\frac{1}{(1+|q|^m)^{2/m}} \leq |f'(q)| \leq \frac{1}{(1-|q|^m)^{2/m}}, \quad \forall q \in \mathbb{B}.$$

*Proof.* Denote the slice regular function

$$g(q) = f'(q)^{-*} * (qf''(q)), \quad \forall q \in \mathbb{B}.$$

From the assumption of  $f$ , we have

$$\operatorname{Re}\left(f'(q)^{-1} qf''(q)\right) > -1, \quad \forall q \in \mathbb{B},$$

then, by Proposition 3.6,  $g$  is slice regular on  $\mathbb{B}$  with  $\operatorname{Reg}(g) > -1$  for all  $q \in \mathbb{B}$ . Then

$$(5.6) \quad |(2+g(q))^{-1}g(q)| < 1, \quad \forall q \in \mathbb{B}.$$

Consider the slice regular function

$$h(q) = (2+g(q))^{-*} * g(q) = (2f'(q) + qf''(q))^{-*} * (qf''(q)), \quad \forall q \in \mathbb{B},$$

with  $h(0) = h'(0) = \dots = h^{(m-1)}(0) = 0$  and  $|h(q)| < 1$  on  $\mathbb{B}$  by Proposition 3.6 again and (5.6).

Whence the Schwarz lemma for slice regular functions in Theorem 3.10 implies that  $|h(q)| \leq |q|^m$  for all  $q \in \mathbb{B}$ . Then

$$|h \circ T_{2+g^c}(q)| \leq |T_{2+g^c}(q)|^m = |q|^m, \quad \forall q \in \mathbb{B}.$$

That is, by Proposition 2.7,

$$|(2+g(q))^{-1}g(q)| \leq |q|^m, \quad \forall q \in \mathbb{B},$$

equivalently,

$$\left|g(q) - \frac{2r^{2m}}{1-r^{2m}}\right| \leq \frac{2r^m}{1-r^{2m}}, \quad |q| = r < 1,$$

which implies that

$$\left|g \circ T_{(f')^c}(q) - \frac{2r^{2m}}{1-r^{2m}}\right| \leq \frac{2r^m}{1-r^{2m}}, \quad |q| = r < 1,$$

i.e., by Proposition 2.7,

$$\left|f'(q)^{-1} qf''(q) - \frac{2r^{2m}}{1-r^{2m}}\right| \leq \frac{2r^m}{1-r^{2m}}, \quad |q| = r < 1.$$

Specially,

$$\left| \operatorname{Re} \left( f'(q)^{-1} q f''(q) \right) - \frac{2r^{2m}}{1-r^{2m}} \right| \leq \frac{2r^m}{1-r^{2m}}, \quad |q| = r < 1.$$

This together with (3.6) yields that

$$-\frac{2r^{m-1}}{1+r^m} \leq \frac{\partial}{\partial r} \log |f'(ru)| \leq \frac{2r^{m-1}}{1-r^m}, \quad q = ru.$$

Integrating the above inequality from 0 to  $r$  gives

$$-\frac{2}{m} \log(1+r^m) \leq \log |f'(ru)| \leq -\frac{2}{m} \log(1-r^m), \quad q = ru.$$

as desired.  $\square$

From Theorem 5.2 we deduce a growth theorem for  $\mathcal{S}^*$ .

**Theorem 5.3.** *Let  $f(q) = q + \sum_{n=m+1}^{+\infty} q^n a_n \in \mathcal{S}^*$  for some positive integer  $m$ . Then*

$$\frac{|q|}{(1+|q|^m)^{2/m}} \leq |f(q)| \leq \frac{|q|}{(1-|q|^m)^{2/m}}, \quad \forall q \in \mathbb{B}.$$

In fact, estimates on the right side of (5.1) and (5.2) can be directly obtained from coefficient estimates in Theorem 1.2. Equality in (5.1) or (5.2) occurs for some  $q_0 \in \mathbb{B} \setminus \{0\}$ , if and only if

$$f(q) = q(1-qu)^{-*2}, \quad q \in \mathbb{B},$$

for some  $u \in \partial\mathbb{B}$ .

Moreover, those two estimates hold in a larger class. Write  $f \prec_{\mathcal{N}} g$  if there exists  $w \in \mathcal{N}(\mathbb{B})$  with  $w(0) = 0$  and  $|w(q)| < 1$ , for all  $q \in \mathbb{B}$  such that  $f(q) = g(w(q))$  for all  $q \in \mathbb{B}$  (see [10, Definition 2.10]). Note that the composition of slice regular functions, when defined, does not give generally a slice regular function. However, if  $w \in \mathcal{N}(\mathbb{B})$ , then the composition  $f \circ w$  is slice regular for any slice regular function  $f$ . Since the proof of Theorem 5.4 below is similar to the classical case (see e.g. [5, p. 202]), we omit it here.

**Theorem 5.4.** *Let  $f \in \mathcal{C}$  and  $g \prec_{\mathcal{N}} f$ . Then the following inequalities hold for all  $q \in \mathbb{B}$*

$$|g(q)| \leq \frac{|q|}{(1-|q|)^2} \quad \text{and} \quad |g'(q)| \leq \frac{1+|q|}{(1-|q|)^3}.$$

In fact, we can obtain more information related to (5.1). Note that in [18] the author established the Hayman's regularity theorem making use of complex version of Theorem 5.5 for the  $p$ -univalent holomorphic function.

**Theorem 5.5.** *Let  $f \in \mathcal{S}^*$  and  $M_{\infty}(r, f) = \max_{|q|=r} |f(q)|$  for  $r \in (0, 1)$ . Then the function  $\phi(r) = \frac{1}{r}(1-r)^2 M_{\infty}(r, f)$  is decreasing on  $(0, 1)$  and hence tends to a limit  $\alpha \in [0, 1]$ . For any  $\alpha \in [0, 1]$ , there exists  $u \in \partial\mathbb{B}$  such that*

$$\lim_{r \nearrow 1^-} (1-r)^2 |f(ru)| = \alpha.$$

*Proof.* From the proof in Theorem 5.1, we have

$$(5.7) \quad \frac{|f(r_2 u)|}{|f(r_1 u)|} \leq \frac{r_2(1-r_1)^2}{r_1(1-r_2)^2}.$$

Choose  $u \in \partial\mathbb{B}$  such that  $M_{\infty}(r_2, f) = |f(r_2 u)|$ . Whence the above inequality yields that

$$\frac{1}{r_2}(1-r_2)^2 M_{\infty}(r_2, f) \leq \frac{1}{r_1}(1-r_1)^2 |f(r_1 u)| \leq \frac{1}{r_1}(1-r_1)^2 M_{\infty}(r_1, f),$$



that is to say  $\phi(r)$  decreases on  $(0, 1)$  and hence tends to a limit  $\alpha$ . From (5.1), there holds that  $\phi(r) \in [0, 1]$ , thus  $\alpha \in [0, 1]$ .

Let  $\{r_n\}$  be a sequence increasing to 1 and choose  $u_n \in \partial\mathbb{B}$  such that  $M_\infty(r_n, f) = |f(r_n u_n)|$ . From the compactness of  $\partial\mathbb{B}$ , there exists a cluster point  $u_\infty \in \partial\mathbb{B}$ . From (5.7), we have, for  $r < r_n$ ,

$$\alpha \leq \frac{1}{r_n}(1 - r_n)^2 |f(r_n u_n)| \leq \frac{1}{r}(1 - r)^2 |f(r u_n)|.$$

Letting  $n \rightarrow \infty$ , we obtain

$$\alpha \leq \frac{1}{r}(1 - r)^2 |f(r u_\infty)| \leq \frac{1}{r}(1 - r)^2 M_\infty(r, f),$$

as desired.  $\square$

As an application of (5.1) in Theorem 5.1, the one-quarter covering theorem for the class  $\mathcal{S}^*$  is obtained. Denote  $B(q_0, r) = \{q \in \mathbb{H} : |q - q_0| < r\}$ .

**Theorem 5.6.** *Let  $f \in \mathcal{S}^*$ , then it holds that*

$$B(0, \frac{1}{4}) \subset f(\mathbb{B}).$$

*The estimate is precise.*

*Proof.* The first inequality in (5.1) implies that

$$(5.8) \quad \liminf_{q \rightarrow \partial\mathbb{B}} |f(q)| \geq \frac{1}{4}.$$

From the open mapping theorem (see [15, Theorem 7.7]), the image set  $f(\mathbb{B})$  is open containing the origin point 0. This together with (5.8) and [30, Proposition 3.1] shows that  $f(\mathbb{B})$  contains the ball  $B(0, 1/4)$ .

The preciseness of estimate can be shown by the slice regular function

$$f(q) = q(1 - q)^{-2}, \quad \forall q \in \mathbb{B}.$$

Then

$$\operatorname{Re}(f(q)^{-1} q f'(q)) = \operatorname{Re}((1 + q)(1 - q)^{-1}) = \frac{1 - |q|^2}{|1 - q|^2} > 0, \quad \forall q \in \mathbb{B},$$

which shows that  $f \in \mathcal{S}^*$ . It is easy to show that  $f(\mathbb{B}) = \{q \in \mathbb{H} : q \notin (-\infty, -1/4]\}$  contains  $B(0, 1/4)$  while contains no ball centred at 0 with radius larger than  $1/4$ , as desired.  $\square$

*Proof of Theorem 1.7.* Let  $f$  be as described in the theorem and assume that  $f(0) = 0$  for otherwise we can consider the slice regular function  $f - f(0)$ . Let  $p \in \partial f(\mathbb{B})$  be a point at minimum distance from the origin 0. If  $|p| = +\infty$ , the theorem has been proved. Otherwise,  $|p| < +\infty$ , we obtain that

$$(5.9) \quad \operatorname{Re}(f(q)\overline{p}) < |p|^2, \quad \forall q \in \mathbb{B},$$

since  $f(\mathbb{B})$  is convex.

Consider the slice regular function

$$g(q) = (2|p|^2 - f(q)\overline{p})^{-*} * f(q)\overline{p}, \quad \forall q \in \mathbb{B}.$$

Then it follows that  $g(0) = 0$  and  $g'(0) = (2p)^{-1}$ . From (5.9), we have

$$|f(q)\overline{p}| < |2|p|^2 - f(q)\overline{p}|, \quad \forall q \in \mathbb{B},$$

which implies, by (3.4) in Proposition 3.6,  $|g(q)| < 1$  for all  $q \in \mathbb{B}$ . The Schwarz lemma for slice regular functions in Theorem 3.10 implies that  $|g'(0)| \leq 1$ . Hence,  $|p| \geq 1/2$ , which shows the image  $f(\mathbb{B})$  contains the open ball  $B(0, 1/2)$ . To see that the constant  $1/2$  is optimal, we consider the slice regular function given by

$$f(q) = q(1 - q)^{-1}, \quad \forall q \in \mathbb{B}.$$

It is easy to show that  $f'(0) = 1$  and  $f(\mathbb{B}) = \{q \in \mathbb{H} : \operatorname{Re} q > -1/2\}$  is convex and contains the open ball centred at 0 of radius  $1/2$  while  $1/2$  is optimal, as desired.  $\square$

Finally, we give a Bloch theorem for convex slice regular functions as a consequence of Theorem 1.7.

Let us introduce some notation. For  $a \in \mathbb{B}$  and the slice regular function  $f$  on  $\mathbb{B}$  with  $f'(0) = 1$ , denote by  $r(a, f)$  the radius of the largest ball contained in the image of  $f$  centred at  $f(a)$ . Define

$$C = \inf\{r(f) : f \text{ is slice regular on } \mathbb{B} \text{ such that } f'(0) = 1 \text{ and } f(\mathbb{B}) \text{ is convex}\}.$$

Now the Bloch theorem for convex slice regular functions can be established as follows.

**Theorem 5.7.**  $\frac{1}{2} \leq C \leq \frac{\pi}{4}$ .

*Proof.* From Theorem 1.7, it suffices to show that  $C \leq \pi/4$ . To this end, let us consider the slice regular function

$$f(q) = \sum_{n=0}^{+\infty} \frac{q^{2n+1}}{2n+1}, \quad q \in \mathbb{B},$$

with  $f'(0) = 1$ .

Note that the image  $f(\mathbb{B}) = \{q \in \mathbb{H} : |\operatorname{Im} q| < \frac{\pi}{4}\}$  is convex, then the desired result follows.  $\square$

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